THE ELECTRIC FIELD IN A CIRCULAR SEMICONDUCTOR PLATE PLACED IN A MAGNETIC FIELD

P. I. Baranskii and Yu. P. Emets

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 7, No. 5, pp. 64–72, 1966

The current distribution in an isothermic isotropically conducting plate of circular form is investigated theoretically and experimentally, in the absence and in the presence of an external magnetic field that is perpendicular to the plate. The general solution of the Riemann-Hilbert boundary value problem has been obtained under these conditions. The analysis of this solution points to experimental possibilities of determining parameters of a crystal under consideration such as the specific electric conductivity (in the absence and in the presence of an external magnetic field), the mobility of current carriers in it, and others.

All the basic results of the calculations undertaken were experimentally verified and quantitatively confirmed in a series of tests carried out on homogeneous monocrystalline n-germanium (with the specific resistivity of 1.1 ohm cm) at room temperature.

It is known that investigations into the galvanomagnetic phenomena (longitudinal and transverse magneto-resistance, the usual, planar and longitudinal Hall effects and others) at the present time constitute not only a means of determining the characteristics of the parameters of the crystals in question (concentration of current carriers, their mobility, etc.) [1], but serve also as a proven and simple means of obtaining important information about the zone structure of crystals [2–5].

Such broadening of the circle of problems affecting the sphere of galvanomagnetic investigations already begins not to correspond to the established traditions of carrying out these investigations on test pieces of rectangular shape (as a rule, in the form of parallelepipeds). This lack of correspondence is greater due to a number of completely logical causes, certain requirements as to the geometrical dimensions of such test pieces (the ratio of length to width) [6] can far from always be satisfied. We note in this connection that in the study of galvanomagnetic phenomena in impulsive magnetic fields, for example, the use of test pieces of circular form would simplify the use of working volumes of small diameter. This, in the final analysis, is equivalent to broadening the scale of magnetic fields that can be used.

The replacement of a rectangular plate by a circular disc enables us also to simplify a measurement of the parameters of semiconductor crystals which usually are obtained in circular form.

Below we present theoretical and experimental investigations into the problem of measuring the galvanomagnetic effects in conducting crystals having a circular form.

§1. THE GENERAL SOLUTION OF THE PROBLEM OF ELECTRICAL CURRENT DISTRIBUTION IN A CIRCULAR PLATE

Let a finite number of electrodes be fixed to the side surface of a plate of thickness h and radius r. The subtended angles and relative location of these electrodes are not fixed beforehand (Fig. 1a). An external magnetic field $H(0, 0, H_z)$ is homogeneous in the plate and is oriented normally to it. The currents to be determined in the plate are assumed to be small, such that their own magnetic field is small in comparison with the external field and can be neglected.

For an isotropic medium in an isothermic case, on the basis of the phenomenological theory we have the following relations between the density of the electric current and the electrostatic potential [7]:

$$j_\rho(\rho, \theta) = -\sigma_{\rho\rho}(H)\frac{\partial\varphi}{\partial\rho} - \sigma_{\rho\theta}(H)\frac{1}{\rho}\frac{\partial\varphi}{\partial\theta}$$

$$j_\theta(\rho, \theta) = -\sigma_{\theta\rho}(H)\frac{\partial\varphi}{\partial\rho} - \sigma_{\theta\theta}(H)\frac{1}{\rho}\frac{\partial\varphi}{\partial\theta},$$

$$H \equiv H_z. \tag{1.1}$$

Here j_ρ, j_θ are the components of the current density vector in a cylindrical coordinate system; $\varphi(\rho, \theta)$ is the potential of the electrostatic field; $\sigma_{\rho\rho}(H)$, $\sigma_{\rho\theta}(H)$, ... are the components of the conductivity tensor.

Using the current continuity equation and the symmetry condition of the coefficients in Eq. (1.1), we can show that, for the assumptions adopted, the potential $\varphi(\rho,\theta)$ is a harmonic function and is determined from the boundary conditions (motion counterclockwise is assumed as the positive direction of the circulation)

$$\sigma_{\rho\rho}(H)\frac{\partial\varphi}{\partial\rho} + \sigma_{\rho\theta}(H)\frac{1}{r}\frac{\partial\varphi}{\partial\theta} = 0 \text{ on } b_k a_{k+1} \quad \begin{pmatrix} k = 1, \ldots, p \\ a_{p+1} = a_1 \end{pmatrix}$$
$$\frac{\partial\varphi}{\partial\theta} = 0 \qquad \text{on } a_k b_k. \tag{1.2}$$

The condition (1.2) is formulated for the field current; it is assumed that recombination is insignificant and that traps for current carriers are absent on the surface of the plate. The first condition signifies that the normal component of the current between electrodes is zero. The second condition is valid if the specific electric conductivity of the electrode is many times larger than $\sigma_{\theta\theta}(H)$ of the material of the plate. Then the arcs $a_k b_k$ (Fig. 1, a) will be equipotential. In a majority of cases of practical importance these conditions are well satisfied. Solving the boundary value problem Eqs. (1.2) for $\varphi(\rho, \theta)$ we can find the value of the overall current, I_k, flowing through the electrodes, and the distribution of the potential on the boundary of the plate from the formulas

$$I_k = rh\int_{\theta_k^a}^{\theta_k^b} j_\rho(r, \theta)\,d\theta = rh\sigma_{\rho\rho}(H)\int_{\theta_k^a}^{\theta_k^b}\left(\frac{\partial\varphi}{\partial\rho}\right)_{\rho=r} d\theta,$$

$$\varphi_\theta = \varphi_k{}^b + \int_{\theta_k^b}^{\theta}\left(\frac{\partial\varphi}{\partial\theta}\right)_{\rho=r} d\theta$$

$$(k=1, \ldots, p). \tag{1.3}$$

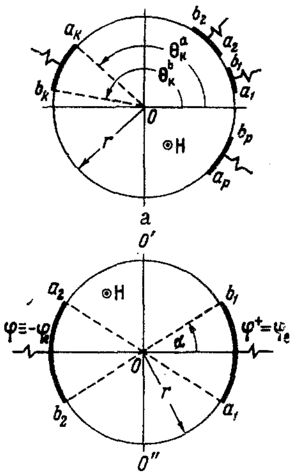

Fig. 1. a) The diagram of a disk with electrodes used to solve the boundary value problem in the general case; b) the diagram of a disk with one pair of symmetrically located electrodes with the subtended angle equal to 2α.

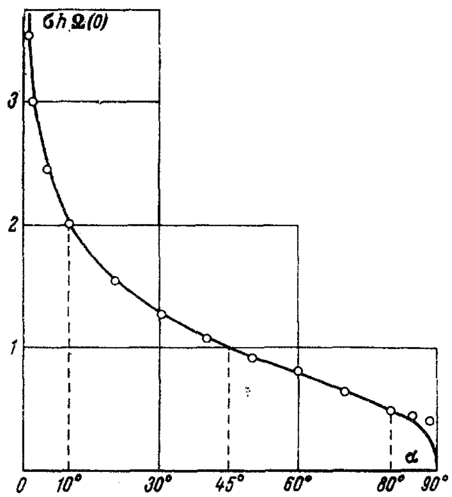

Fig. 2. The relation between $h\sigma\Omega(0)$ and the angle α. Circles indicate the experimental results. The calculated results, obtained from the formula (2.9), are marked by the solid line.

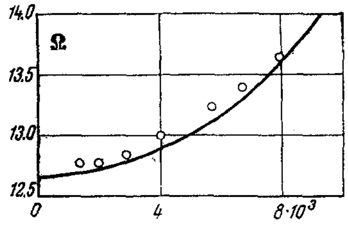

Fig. 3. The theoretical curve of the relation between Ω and H for $\alpha = \pi/4$, obtained from the first formula (2.13) for the case where current carriers are absorbed in lattice oscillations. Circles denote the results of an experimental check of the relation between Ω and H for the same conditions.

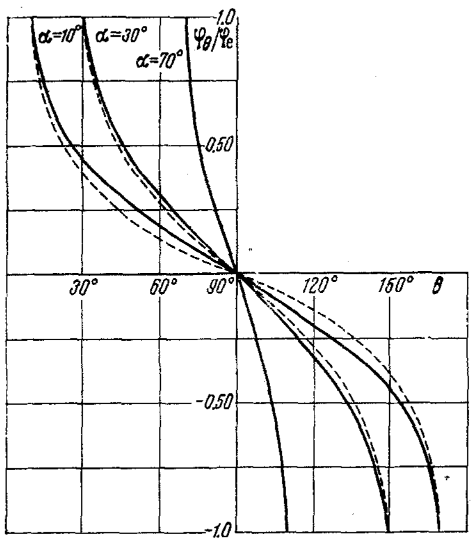

Fig. 4. The distribution of electric potentials on the periphery of the disk for H = 0 and α = 10, 30 and 70°, calculated from the formula (2.17), is shown by the solid lines. The results of the corresponding experiments are shown by dashed lines.

The function

$$W(z) = U(\rho, \theta) + iV(\rho, \theta) = \rho\,\frac{\partial\varphi}{\partial\rho} - i\,\frac{\partial\varphi}{\partial\theta} \qquad (z = \rho e^{i\theta}) \qquad (1.4)$$

holomorphic in a circle, is brought into the analysis.

To find W(z) we obtain the Riemann-Hilbert boundary value problem with discontinuous coefficients from Eq. (1.2):

$$\sigma_{\rho\rho}(H)\,U(r,\theta) - \sigma_{\rho\theta}(H)\,V(r,\theta) = 0$$

$$\text{for } z = re^{i\theta} \text{ for } \theta_k{}^b < \theta < \theta_{k+1}^a,$$

$$V(r,\theta) = 0 \text{ for } z = re^{i\theta} \text{ for } \theta_k{}^a < \theta < \theta_k{}^b,$$

$$W(0) = 0 \quad (k = 1, \ldots, p;\ \theta_{p+1}^a = \theta_1{}^a). \qquad (1.5)$$

Instead of (1.3) we have

$$I_k = h\sigma_{\rho\rho}(H) \int\limits_{\theta_k{}^a}^{\theta_k{}^b} U(r,\theta)\,d\theta,$$

$$\varphi_\theta = \varphi_k{}^b - \int\limits_{\theta_k{}^b}^{\theta} V(r,\theta)\,d\theta \qquad (k = 1, \ldots, p). \qquad (1.6)$$

It follows from these formulas and the condition (1.5) that I_k and φ_θ do not depend on the radius r of the plate; they depend on the dimensions of the electrodes, their relative location and the plate thickness h. Taking into account this circumstance, we take the radius of the circle to be equal to 1. The function W(z), by means of inversion, is analytically continued through the arc $a_k b_k$ (k = 1, . . . , p) in the interior of a unit circle. Then instead of the problem (1.5) we arrive at the generalized problem of linear union (Riemann's problem) [8, 9], which can be solved more simply

$$\Psi^+(t) = -\frac{\sigma_{\rho\rho}(H) - i\sigma_{\rho\theta}(H)}{\sigma_{\rho\rho}(H) + i\sigma_{\rho\theta}(H)}\,\Psi^-(t)$$

$$\text{for } t = e^{i\theta} \text{ for } \theta_k{}^b < \theta < \theta_{k+1}^a,$$

$$\Psi^+(t) = \Psi^-(t) \text{ for } t = e^{i\theta} \text{ for } \theta_k{}^a < \theta < \theta_k{}^b, \qquad (1.7)$$

$$\Psi(z) = \begin{cases} \Psi^+(z) \text{ for } |z| < 1, \\ \quad \Psi^+(z) = W(z) = U(\rho,\theta) + iV(\rho,\theta) \\ \Psi^-(z) \text{ for } |z| > 1, \\ \quad \Psi^-(z) = \overline{W(1/\bar{z})} = \\ \quad = U(1/\rho, -\theta) - iV(1/\rho - \theta). \end{cases} \qquad (1.8)$$

At the points z = 0 and z = ∞ the piecewise holomorphic function Ψ(z) is characterized by the expansions

$$\Psi^+(z) = A_1{}^0 z + A_2{}^0 z^2 + \ldots,$$

$$\Psi^-(z) = \frac{B_1{}^0}{z} + \frac{B_2{}^0}{z^2} + \ldots, \qquad (1.9)$$

as follows from the third condition (1.5) and (1.8).

The solution of the problem (1.7)–(1.9) is constructed in a class of functions which have integrable singularities at the points a_k, b_k (k = 1, . . . , p). The latter point to an increased density of the electric current in the neighborhood of the ends of the electrodes.

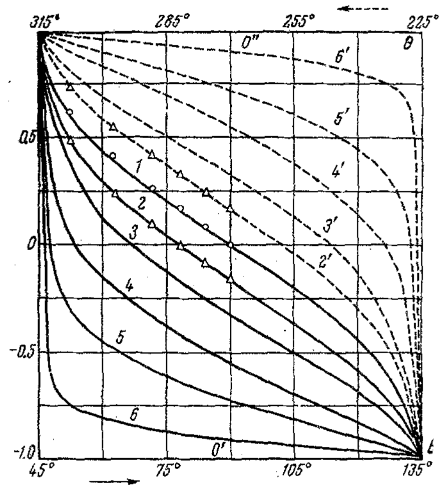

Fig. 5. The potential distributions on the circumference of the disk for various values of the parameter χ for α = π/4: 1) χ = 0, 2) χ = 0.25, 3) χ = 0.5, 4) χ = = 1.0, 5) X = 2.0, 6) χ = 6.3.

The function Ψ(z) vanishes at infinity, and at the point z = 0 it assumes the zero value

$$\Psi(z) = \prod_{k=1}^{p} \frac{[z - \exp(i\theta_k{}^a)]^{-1/2-\varepsilon}}{[z - \exp(i\theta_k{}^b)]^{1/2-\varepsilon}} \left\{ \prod_{k=1}^{p} \times \right.$$

$$\left. \times \exp\left[-\frac{i}{2}(\theta_k{}^a + \theta_k{}^b) - i\varepsilon(\theta_k{}^a - \theta_k{}^b)\right] \right\}^{-1/2} P(z), \qquad (1.10)$$

$$P(z) = C_1 z^{p-1} + C_2 z^{p-2} + \ldots + C_{p-1} z. \qquad (1.11)$$

Here

$$\varepsilon = \frac{1}{\pi} \operatorname{arc\,tg} \frac{\sigma_{\rho\theta}(H)}{\sigma_{\rho\rho}(H)} = \frac{1}{\pi} \operatorname{arc\,tg} R_H \sigma(H) H$$

$$\left(0 \leqslant \varepsilon < \frac{1}{2}\right)$$

$$R_H = \frac{\sigma_{\rho\theta}(H)}{H\,[\sigma_{\rho\rho}{}^2(H) + \sigma_{\rho\theta}{}^2(H)]},$$

$$\sigma(H) = \frac{\sigma_{\rho\rho}{}^2(H) + \sigma_{\rho\theta}{}^2(H)}{\sigma_{\rho\theta}(H)}. \qquad (1.12)$$

The constants $C_k = A_k' + iB_k'$ (k = 1, . . . , p) in Eq. (1.11) are connected by the relations

$$A_{p-k-1}' + iB_{p-k-1}' = A_k' - iB_k'. \qquad (1.13)$$

In the formula (1.10) the root is taken with the positive sign, and under the function

$$X(z) = \prod_{k=1}^{p} [z - \exp(i\theta_k{}^a)]^{-1/2-\varepsilon} [z - \exp(i\theta_k{}^b)]^{-1/2+\varepsilon}$$

implies a curve which for large |z| assumes the form

$$X(z) = \frac{1}{z^p} + \frac{\Upsilon_{-p-1}}{z^{p+1}} + \ldots \qquad (|z| \to \infty). \qquad (1.14)$$

Equations (1.1), expressing the generalized Ohm's law, are written in the complex form

$$j(z) = j_\rho(\rho, \theta) - ij_\theta(\rho, \theta) =$$

$$= -[\sigma_{\rho\rho}(H) - i\sigma_{\rho\theta}(H)]\left(\frac{\partial\varphi}{\partial\rho} - i\frac{1}{\rho}\frac{\partial\varphi}{\partial\theta}\right). \qquad (1.15)$$

In obtaining Eq. (1.15) we have used the condition of symmetry of the conductivity tensor in an isotropic

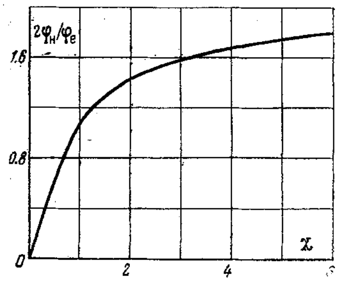

Fig. 6. The theoretical relation $2\varphi_H/\varphi_e = f(\chi)$.

medium

$$\sigma_{\rho\theta}(H) = -\sigma_{\theta\rho}(H), \qquad \sigma_{\rho\rho}(H) = \sigma_{\theta\theta}(H). \qquad (1.16)$$

Taking into account the Eqs. (1.4) and (1.8), we obtain the sought for current distribution in the circular plate

$$j(z) = j_\rho - ij_\theta = -\frac{1}{\rho}[\sigma_{\rho\rho}(H) - i\sigma_{\rho\theta}(H)]\Psi^+(z) \qquad (1.17)$$

or

$$\overline{j(z)} = j_\rho + ij_\theta = -\frac{1}{\rho}[\sigma_{\rho\rho}(H) + i\sigma_{\rho\theta}(H)]\overline{\Psi^+(z)}\cdot \qquad (1.18)$$

The constants C_k remain to be determined. For this we must enlist additional boundary conditions. For this it is sufficient to specify either the potentials on the electrodes or the values of the currents flowing through them, or to specify the potentials on certain electrodes and the currents in the others. Let, for example, the values of the currents $I_k (k = 1, \ldots, p)$ be given. The normal component of the current in the electrodes is given by the formula

$$j_\rho(1, \theta) = -\sigma_{\rho\rho}(H)\Psi^+(\theta)\cdot \qquad (1.19)$$

Here, as is easily shown,

$$\Psi^+(\theta) = T\,[A_1' \cos(^1/_2 p - 1)\theta -$$

$$- B_1' \sin(^1/_2 p - 1)\theta + A_2' \cos \times (^1/_2 p - 2)\theta -$$

$$- B_2' \sin(^1/_2 p - 2)\theta + \ldots + {}^1/_2 A'_{^1/_2 p}],$$

$$T^{-1} = 2\prod_{k=1}^{p}\left(\sin\frac{\theta - \theta_k^a}{2}\right)^{1/2+\varepsilon}\left(\sin\frac{\theta - \theta_k^b}{2}\right)^{1/2-\varepsilon}. \qquad (1.20)$$

The constants A_1', B_1', A_2', B_2', ... are determined from the solution of the system

$$I_k = h\int_{\theta_k^a}^{\theta_k^b} j_\rho(1, \theta)\, d\theta =$$

$$= -h\sigma_{\rho\rho}(H)\int_{\theta_k^a}^{\theta_k^b}\Psi^+(\theta)\, d\theta \qquad (k = 1, \ldots, p). \qquad (1.21)$$

We conclude by finding the unknown constants of the general solution of the problem of current distribution in a circular plate.

§2. A PARTICULAR CASE: A PLATE WITH ONE PAIR OF ELECTRODES

As an application of the general theory, let us consider the galvanomagnetic phenomena in a semiconductor plate with one pair of symmetrically located electrodes (Fig. 1b). To fix ideas we take a nondegenerate n-type semiconductor having a simple zone structure and an arbitrary absorption mechanism of current carriers.

To obtain concrete values for the coefficients in Eq. (1.1), we must turn to the kinetic equation of Boltzmann, whence we obtain the equation for the current [10]

$$\mathbf{j} = \frac{ne^2}{m}\left\langle\frac{\tau}{1 + (e\tau H/mc)^2}\right\rangle\mathbf{E} -$$

$$-\frac{ne^3}{m^2c}\left\langle\frac{\tau^2}{1 + (e\tau H/mc)^2}\right\rangle\mathbf{E}\times\mathbf{H}. \qquad (2.1)$$

Here e, m and n are the charge, mass and concentration of electrons; τ is the relaxation time; H is the magnetic field intensity vector; E is the electric field intensity vector; c is the velocity of light.

The symbol $\langle g\rangle$ denotes the averaging integral

$$\langle g\rangle = \frac{4}{3\sqrt{\pi}}\int_0^\infty g(x)\, x^{3/2}e^{-x}\, dx \qquad \left(x = \frac{\varepsilon_0}{K_0T}\right).$$

Its numerical value depends on the absorption mechanism of electrons in the semiconductor, where ε_0, T are the energy and temperature of the electrons, and k_0 is Boltzmann's constant. Decomposing Eq. (2.1) along the appropriate axes of the cylindrical coordinate system, we obtain

$$j_\rho(\rho, \theta) = -\lambda_1\frac{\partial\varphi}{\partial\rho} + \lambda_2\frac{1}{\rho}\frac{\partial\varphi}{\partial\theta},$$

$$j_\theta(\rho, \theta) = -\lambda_2\frac{\partial\varphi}{\partial\rho} - \lambda_1\frac{1}{\rho}\frac{\partial\varphi}{\partial\theta},$$

$$\lambda_1 = \frac{ne^2}{m}\left\langle\frac{\tau}{1 + (e\tau H/mc)^2}\right\rangle,$$

$$\lambda_2 = \frac{ne^3H}{m^2c}\left\langle\frac{\tau^2}{1 + (e\tau H/mc)^2}\right\rangle. \qquad (2.2)$$

Following the results of the preceding section, for the given case we have

$$\Psi(z) = \frac{Cz}{\sqrt{\exp(-4i\alpha\varepsilon)}} (z^2 - e^{-i2\alpha})^{-1/2+\varepsilon} (z^2 - e^{i2\alpha})^{-1/2-\varepsilon},$$

$$\Psi^+(z) = W(z) \quad \text{for } |z|<1,$$

$$j(z) = j = ij_\theta = -\frac{1}{\rho}(\lambda_1 - i\lambda_2)\times$$

$$\times\left(\rho\frac{\partial\varphi}{\partial\rho} - i\frac{\partial\varphi}{\partial\theta}\right) = -\frac{1}{\rho}(\lambda_1 - i\lambda_2)\Psi^+(z),$$

$$\varepsilon = \frac{1}{\pi}\operatorname{arc\,tg}\frac{\lambda_2}{\lambda_1} = \frac{1}{\pi}\operatorname{arc\,tg} R_H\sigma(H)H, \quad 0 \leqslant \varepsilon < \frac{1}{2},$$

$$R_H = \frac{-\lambda_2}{H(\lambda_1^2+\lambda_2^2)}, \qquad \sigma(H) = -\frac{\lambda_1^2+\lambda_2^2}{\lambda_1}. \quad (2.3)$$

Using the formulas of Sokhotskii-Plemel, after a number of transformations we find the difference of potentials between the electrodes $2\varphi_e$, the overall current I flowing through the electrodes and the plate resistance Ω. These integral characteristics of the plate in the general case depend on the physical parameters λ_1, λ_2 and the geometrical quantities α and h

$$2\varphi_e(\lambda_1, \lambda_2, \alpha) = \frac{\lambda_1 C_1 N(\alpha, \varepsilon)}{\sqrt{\lambda_1^2+\lambda_2^2}}, \quad (2.4)$$

$$I(\lambda_1, \lambda_2, \alpha, h) = \lambda_1 h C_1 Q(\alpha, \varepsilon), \quad (2.5)$$

$$\Omega(\lambda_1, \lambda_2, \alpha, h) = \frac{2\varphi_e}{I} = \frac{1}{h\sqrt{\lambda_1^2+\lambda_2^2}}\frac{N(\alpha, \varepsilon)}{Q(\alpha, \varepsilon)}, \quad (2.6)$$

where

$$N(\alpha, \varepsilon) = \int_{\pi-\alpha}^{\alpha} [\sin(\theta+\alpha)]^{-1/2+\varepsilon}[\sin(\theta-\alpha)]^{-1/2-\varepsilon}\, d\theta, \quad (2.7)$$

$$Q(\alpha, \varepsilon) =$$

$$= \int_{\pi-\alpha}^{\pi+\alpha} [\sin(\theta+\alpha-\pi)]^{-1/2+\varepsilon}[\sin(\theta-\alpha)]^{-1/2-\varepsilon}\, d\theta. \quad (2.8)$$

A. 1°. Let us analyze the formula (2.6) in the absence of the magnetic field

$$\Omega(\sigma, \alpha, h) = \frac{1}{\sigma h}\frac{N(\alpha, 0)}{Q(\alpha, 0)} =$$

$$= \frac{1}{\sigma h}\frac{K(\cos\alpha)}{K(\sin\alpha)} \quad \left(\sigma = \frac{ne^2\langle\tau\rangle}{m}\right). \quad (2.9)$$

Here σ is the specific electric conductivity of the semiconductor, K(k) is a complete elliptic integral of the first kind. The connection established here between the plate resistance Ω and the conductivity σ assumes a particularly simple form for $\alpha = \pi/4$:

$$\Omega(\sigma, h) = \Omega(0) = \frac{1}{\sigma h} = \frac{m}{ne^2\langle\tau\rangle h}. \quad (2.10)$$

2°. However, if $H \neq 0$, then difficulties arise in the case of arbitrary values of α in calculating the integrals according to the formula (2.6). The expression for the plate resistance Ω assumes the following simple form for $H \neq 0$, when $2\alpha = \pi/2$

$$\Omega(\lambda_1, \lambda_2, h) = \frac{1}{h\sqrt{\lambda_1^2+\lambda_2^2}}\frac{B[1/4(1-2\varepsilon), 1/4(1+2\varepsilon)]}{B[1/4(1+2\varepsilon), 1/4(1-2\varepsilon)]}. \quad (2.11)$$

Here B(p, q) is the beta function. Taking into account the symmetry of the beta function with respect to its parameters B(p, q) = B(q, p), we finally obtain

$$\Omega(\lambda_1, \lambda_2, h) = \frac{1}{h\sqrt{\lambda_1^2+\lambda_2^2}} = \frac{\cos\pi\varepsilon}{\lambda_1 h}. \quad (2.12)$$

The formula (2.12) is the exact solution of the original boundary value problem for $\alpha = \pi/4$; it is valid for any magnetic field (not yet leading to quantization). We also refer to the two approximate formulas for weak ($\mu H/c \ll 1$; terms of the order H^2 are retained in the expansions λ_1 and λ_2 with respect to H) and strong ($\mu H/c \gg 1$) magnetic fields respectively

$$\Omega(H, \mu, \ldots) = \Omega(H) =$$

$$= \Omega(0)\left[1 - 2\left(\frac{\mu H}{c}\right)^2\frac{\langle\tau^3\rangle}{\langle\tau\rangle^3} + \left(\frac{\mu H}{c}\right)^2\frac{\langle\tau^2\rangle^2}{\langle\tau\rangle^4}\right]^{-1/2}$$

$$\left(\mu = \frac{e}{m}\langle\tau\rangle = \frac{\sigma}{ne}\right)$$

$$\Omega(n, e, H, h) = \frac{H}{nech} = R_{H\infty}\frac{H}{h}. \quad (2.13)$$

Although the plate resistance grows as H increases, the magneto-resistance of the semiconductor, as was to be expected in the framework of the kinetic theory in the case of $H \to \infty$, reaches saturation.

B. 1°. The expression for the potential distribution on the boundary of the plate is obtained from the second condition of Eq. (1.6):

$$\varphi_\theta(\lambda_1, \lambda_2, \alpha, \varphi_e) = \varphi_e + \frac{\lambda_1 C_1}{\sqrt{\lambda_1^2+\lambda_2^2}}\times$$

$$\times\int_\alpha^\theta [\sin(\theta+\alpha)]^{-1/2+\varepsilon}[\sin(\theta-\alpha)]^{-1/2-\varepsilon}\, d\theta. \quad (2.14)$$

The real constant C_1 is determined from Eq. (2.4) or (2.5).

In the absence of a magnetic field Eq. (2.14) is reduced to the form

$$\varphi_0(\alpha, \varphi_e) =$$

$$= \begin{cases} \varphi_e[1 - F(\gamma_1, \cos\alpha)/K(\cos\alpha)] & \text{for } \alpha \leqslant \theta \leqslant 1/2\pi \\ \varphi_e F(\gamma_2, \cos\alpha)/K(\cos\alpha) & \text{for } 1/2\pi \leqslant \theta \leqslant \pi - \alpha \end{cases}$$

$$\gamma_1 = \arccos\frac{\operatorname{tg}\alpha}{\operatorname{tg}\theta}, \qquad \gamma_2 = \operatorname{arc\,tg}\left(\frac{1+\operatorname{tg}^2\alpha}{1+\operatorname{tg}^2\theta}\right)^{1/2}. \quad (2.15)$$

Here $F(\psi, k)$ is an incomplete elliptic integral and K(k) is a complete elliptic integral of the first kind.

It should be pointed out that when H = 0, the original problem of Riemann-Hilbert becomes a mixed problem of the theory of holomorphic functions; its solution is known [8–9, 11]. Therefore the resulting

Eqs. (2.9), (2.10) and (2.15) are also obtained directly by applying the formula of Keldysh-Sedov.

2°. Let us consider now the potential distribution on the boundary of the plate, when $H \neq 0$ and $\alpha = \pi/4$. In the same way as in obtaining the formulas (2.15), we assume that the potentials on the electrodes are given ($\varphi_{a_1b_1} = \varphi_e$, $\varphi_{a_2b_2} = -\varphi_e$; Fig. 1, b), and determine the constant C_1 from the relation (2.4). We then obtain

$$\varphi_\theta(\varepsilon, \varphi_e) = \varphi_e\left[1 - 2I_t\left(\frac{1-2\varepsilon}{4}, \frac{1+2\varepsilon}{4}\right)\right] =$$

$$= \varphi_e\left[1 - \frac{8}{1-2\varepsilon} t^{1/4(1-2\varepsilon)} F \times\right.$$

$$\left.\times\left(\frac{1-2\varepsilon}{4}, \frac{3-2\varepsilon}{4}; \frac{5-2\varepsilon}{4}; t\right)\right],$$

$$I_t\left(\frac{1-2\varepsilon}{4}, \frac{1+2\varepsilon}{4}\right) = \frac{B_t[1/4(1-2\varepsilon), 1/4(1+2\varepsilon)]}{B[1/4(1-2\varepsilon), 1/4(1+2\varepsilon)]}$$

$$\begin{pmatrix} t = \sin^2(\theta - 1/4\pi) \\ 0 \leqslant t \leqslant 1 \end{pmatrix}. \qquad (2.16)$$

Here $B_t(p, q)$, $I_t(p, q)$ are incomplete beta functions, and $B(p, q)$ is a complete beta function; $F(p, 1-q; p+1; t)$ is a hypergeometric series.

For weak and strong magnetic fields the value of the parameter ε, defined by Eq. (2.3), is given respectively by the formulas

$$\varepsilon = \frac{1}{\pi} \operatorname{arc\,tg} \frac{\mu H}{c} \frac{\langle\tau^2\rangle}{\langle\tau\rangle^2},$$

$$\varepsilon = \frac{1}{\pi} \operatorname{arc\,tg} \frac{\mu H}{c} \frac{\langle\tau^{-1}\rangle}{\langle\tau\rangle^{-1}}. \qquad (2.17)$$

3°. EXPERIMENTAL RESULTS AND COMPARISON WITH THE THEORY

The tests to check certain results of the theory presented here were carried out on n-germanium having specific resistivity 1.1 ohm cm and possessing a fairly high degree of homogeneity. The measurements were carried out on a plate with the diameter equal to 29 mm and the thickness h = 0.87 mm, with two symmetrically located electrodes. The plane of the disk coincided with the crystallographic plane (111). The contacts were soldered to the plate with tin containing some antimony; in the range of currents used they were ohmic. All this reliably ensured that the boundary conditions formulated in posing the problem were satisfied. The experiments were carried out at room temperature over an interval of magnetic fields of 1300 < H < 7900 Oe, by the compensation method, for two directions of I and H. The generally adopted precautionary measures for eliminating exposure, effect of convective heat flows and other disturbances were observed.

A. 1°. The relation between the general resistance, Ω, of the plate and the subtending angle of the electrode, 2α, was experimentally verified in the absence of a magnetic field. From the measured data $\Omega(0) = f(\alpha)$ we have constructed the relationship between $\sigma h\,\Omega(0)$ and the angle α, represented by the circles in Fig. 2. In calculating $\sigma h \Omega(0)$ we have used the value of σ measured on a rectangular plate which has afterward cut out from the disk under consideration. The calculated data of the same relation obtained from Eq. (2.9) is shown in Fig. 2 by the continuous line. The test and theoretical data coincide wholly for almost all angles α. This allows us to recommend the formula (2.9) for the determination of the specific magnetic conductivity σ directly from the measured general resistance Ω of the plate. For this it is convenient to use electrodes with the subtended angles α = 10, 45 and 80° for which from Eq. (2.9) we respectively obtain

$$\sigma = \frac{2}{h\Omega(\sigma, h, \alpha = 10°)}, \quad \sigma = \frac{1}{h\Omega(\sigma, h, \alpha = 45°)}$$

$$\sigma = \frac{1}{2h\Omega(\sigma, h, \alpha = 80°)}. \qquad (3.1)$$

2°. It is of interest to measure the general resistance from H. The relation between Ω and H was specifically verified for the case $\alpha = \pi/4$. The calculated results, obtained for the case of lattice absorption from the first formula (2.13) (for the value μ = 3100 cm /V sec), are shown in Fig. 3 by the solid curve; the points here represent the results of the tests carried out. It follows from Fig. 3 that the expression obtained here for Ω (in a quadratic approximation with respect to the magnetic field) agrees well with the test results. Therefore, when working in the region $\mu H/c \ll 1$, the first formula (2.13) together with the Hall effect data can be used to elucidate the absorption mechanism of current carriers. For a known absorption mechanism it can be used to determine the mobility values.

The second formula (2.13) is also of interest for us, from a viewpoint of its practical use. But it is advisable to leave the analaysis of this problem until after this expression has been experimentally verified.

B. 1°. In Fig. 4 the solid lines represent the theoretical potential distributions between the electrodes on the circumference of the disc for H = 0. These were calculated from Eq. (2.15) for three values α = 10, 30, and 70°. In the same figure the experimental data (under the conditions assumed here) are shown by the dashed lines.

The potential distribution predicted by the theory agrees fairly well with the test results. Some slight discrepancies are observed for small values of α, and they are apparently connected with the effect of the small inhomogeneities that exist in any crystal. Their appearance for large disk dimensions and small α seem to be more probable for a number of reasons.

2°. Considerably more interesting theoretically and more important in practice are the potential distributions obtained for various H, which for $\alpha = \pi/4$ are determined by the first expression (2.16). Unfortunately, the incomplete beta functions are not tabulated for the parameter values $p = (1 - 2\varepsilon)/4$ and $q = (1 + 2\varepsilon)/4$ where $0 \leq \varepsilon < 1/2$, and the hypergeometric series in terms of which they are expressed converges very slowly. Therefore the improper integrals determining the beta functions were calculated by the method of excluding singularities proposed by L. V. Kantorovich [12]. The theoretical curves of potential distributions on the circumference of the disk, for various values of the parameter $\chi = R_H\sigma(H)H = \mu_H H/c$, and the experimental data for χ = 0.25 are shown in Fig. 5. The solid lines in this figure correspond to the potential distribution on the arc b_1a_2 (see Fig. 1, b); the dashed lines refer to the arc b_2a_1. The test results agree well with the theory; this enables us to determine experimentally the Hall mobility $\mu_H = \chi c/H$ on test pieces having the form of a disc.

For this it is convenient to measure the Hall voltage between the points 0'0" of the test piece (see Fig. 1, b). From the value of this voltage we find χ by means of the theoretical curve $2\varphi_H/\varphi_e = f(\chi)$ (Fig. 6). The value of μ_H is determined from χ (for a given H). In our tests we obtained $2\varphi_H/\varphi_e = 0.321$, which corresponds to $\chi = 0.25$; $\mu_H = 3180\ cm^2/V$ sec. Measurements of the Hall effect on a rectangular plate cut out from this disk gave $\mu_H = 3100\ cm^2/V$ sec.

In conclusion we note that a different method of calculating the fields in semiconductor plates for certain regions, and recommendations for determining the galvanometric parameters are discussed in [13–15].

The authors thank V. V. Gaiduchenko for his help in carrying out the tests.

REFERENCES

1. A. F. Ioffe, The Physics of Semiconductors [in Russian], Izd-vo AN SSSR, 1957.

2. F. Seitz, "Note on the theory of resistance of a cubic semiconductor in a magnetic field," Phys. Rev., vol. 79, no. 2, 1950.

3. P. I. Baranskii and P. M. Kurilo, "Investigation of the symmetry properties of isoenergic surfaces in n-germanium by means of Hall effect measurements," Fiz. tverd. tela, vol. 6, no. 1, 1964.

4. P. I. Baranskii, I. V. Dakhovskii, and P. M. Kurilo, "The anisotropy of the Hall coefficients of n-silicon in the region of intermediate magnetic fields," Fiz. tverd. tela, vol. 6, no. 7, 1964.

5. P. I. Baranskii, I. V. Dakhovskii, and P. M. Kurilo, "The concentration relationship of anisotropy of the Hall coefficients in n-germanium," Fiz. tverd. tela, vol. 6, no. 10, 1964.

6. I. Isenberg, B. R. Russell, and R. F. Green, "Improved method for measuring Hall coefficients," Rev. Sci. Instr., vol. 19, no. 10, 1948.

7. A. C. Beer, Galvanomagnetic Effects in Semiconductors, Academic Press, New York and London, 1963.

8. N. I. Muskhelishvili, Singular Integral Equations [in Russian], Fizmatgiz, 1962.

9. F. D. Gakhov, Boundary Value Problems [in Russian], Fizmatgiz, 1963.

10. A. I. Ansel'm, Introduction to the Theory of Semiconductors [in Russian], Fizmatgiz, 1962.

11. M. V. Keldysh and L. I. Sedov, "Effective solution of certain boundary value problems for harmonic functions," Dokl. AN SSSR, vol. 16, no. 1, 1937.

12. L. V. Kantorovich, "On the approximate calculation of certain types of definite integrals, and other applications of the method of excluding singularities," Matem. sb., vol. 41, no. 2, 1934.

13. R. F. Wick, "Solution of the field problem of the germanium generator," J. Appl. Phys., vol. 25, no. 6, 1954.

14. H. J. Lippmann and F. Kuhrt, Der Geometrie-Einfluss auf den transversalen magnetischen Widerstandseffekt bei rechteckformigen Halbleiterplatten," Z. Naturforsch., B. 13, 462–474, 1958.

15. L. J. Van der Pauw, "A method of measuring specific resistivity and Hall effect of disks of arbitrary shape," Philips Res. Repts, vol. 13, no. 1, 1958.

27 February 1966 Kiev